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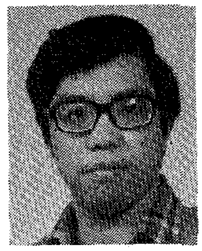

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Restoration of Continuously Sampled Band-Limited Signals from Aliased Data

ROBERT J. MARKS, II, MEMBER, IEEE

Abstract—A continuously sampled signal is obtained by periodically placing a signal to zero. A straightforward closed form method is presented for restoration of continuously sampled bandlimited signals—even when the data is aliased. The sampled signal is simply multiplied by a periodic function specified by the duty cycle of the degradation and the severity of aliasing. This product is then placed through a filter with bandwidth equal to that of the signal. The filter acts as an interpolator and the original signal is restored.

INTRODUCTION

THE restoration problem under consideration is as follows: A bandlimited signal is periodically set to zero. Given the signal's bandwidth, we wish to reconstruct the original signal even when the data are aliased. Such analysis, for example, is useful for the restoration of spatially multiplexed images stored on a continuous medium.

On the surface we are seemingly confronted with a paradox. On one hand, aliased data are commonly assumed to be degraded beyond recovery. On the other hand, knowledge of a *band-limited signal over any arbitrarily small interval is* sufficient to specify the signal everywhere. This follows from well known analyticity arguments [1]-[3]. We will demonstrate that the former assumption is incorrect for the problem at hand. Indeed, a number of well known techniques can be applied to this problem.

One technique involves taking a sample of the sampled signal and its first M-1 derivatives in each known interval. If T is the sample period, then the image can be recovered in M/T exceeds or equals the Nyquist rate [3]-[5].

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A second solution involves application of a sampling theorem using interlaced samples [3], [5]-[6] or, more generally, from irregularly spaced sample points [3], [5]-[10]. As long as there exists at least one sample per Nyquist interval, the signal can be recovered. Clearly, we simply need to sample the object at a sufficiently dense rate over those intervals where the signal is known.

The above restoration schemes require discrete sampling of the degraded signal. Known data are not used. One analog restoration technique requires cumbersome evaluation of an integral equation directly analogous to Slepian and Pollak's classical analysis [1]-[3]. An alternate analog technique is a straightforward modification of Gerchberg's iterative algorithm [3], [11]-[15] which can be placed in closed form [15]-[18].

Continuously sampled signals can also be restored to an approximation by using linear or logarithmic filtering [19]. Indeed, if the sampling rate is sufficiently fast, the data can be unaliased and exact deterministic interpolation is possible in the spirit of the conventional sampling theorem [5]. Rader [20] has presented a restoration technique for undersampled periodic functions.

In this paper, we present an algorithm for restoring continuously sampled band-limited signals from aliased data. The continuously sampled signal is multiplied by a periodic function specified by the severity of aliasing and the periodic degradation's duty cycle. This product is low pass filtered. The result in the absence of noise is the original signal.

Preliminaries

Consider a finite energy band-limited signal $f(x)$ with bandwidth $2W$. That is

$$f(x) = \int_{-W}^{W} F(u) \exp(j2\pi ux)\, du$$

where

$$F(u) = \int_{-\infty}^{\infty} f(x) \exp(-j2\pi ux)\, dx.$$

Define the periodic pulse train with unit period by

$$r_\alpha(x) = \sum_{n=-\infty}^{\infty} \operatorname{rect}\left(\frac{x-n}{\alpha}\right) \tag{1}$$

where

$$\operatorname{rect}(\xi) = \begin{cases} 1; & |\xi| \leq \frac{1}{2} \\ 0; & |\xi| > \frac{1}{2} \end{cases}$$

and

$$\alpha < 1$$

is the pulse train's duty cycle. The continuously sampled band-limited image is defined by

$$g(x) = f(x)\, r_\alpha(x/T) \tag{2}$$

where T is the pulse train's period. The interpolatory restoration problem is to determine $f(x)$ from knowledge of $g(x)$, $r_\alpha(x/T)$ and $2W$.

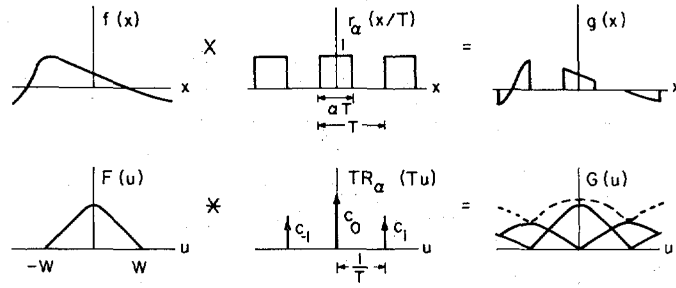

Fig. 1. Illustration of the degradation of $f(x)$ to $g(x)$ (a) in x; (b) in the frequency domain.

The degradation process described by (2) is illustrated by the top three functions in Fig. 1. The corresponding operation in the frequency domain, shown in the bottom three functions in Fig. 1, is

$$G(u) = F(u) * TR_\alpha(Tu) \tag{3}$$

where the upper case letters denote the Fourier transforms of the corresponding functions in (2) and the asterisk denotes convolution. Expanding (1) in a Fourier series followed by transformation gives

$$TR_\alpha(Tu) = \sum_{n=-\infty}^{\infty} c_n \delta\left(u - \frac{n}{T}\right)$$

where

$$c_n = \alpha \operatorname{sinc} \alpha n = c_{-n}$$

and $\operatorname{sinc} \xi = \sin(\pi\xi)/(\pi\xi)$. Thus (3) can be written as

$$G(u) = \sum_{n=-\infty}^{\infty} c_n F\left(u - \frac{n}{T}\right).$$

Clearly, if the sampling rate $1/T$ exceeds $2W$, the replicated spectra do not overlap and $F(u)$ can be regained from $G(u)$ by a simple low pass filter [5]. We are interested in the aliased case. If one spectra overlaps the right half zero order spectra as in Fig. 2(a), we have first order aliasing. If two overlap, as in Fig. 2(b), we have second order aliasing, etc. In general, the order of aliasing is

$$M = \langle 2WT \rangle$$

where $\langle x \rangle$ denotes "the greatest integer less than or equal to x."

A Restoration Technique for First Order Aliasing

The methodology for restoration is best introduced by the first order aliasing interpolation example illustrated in Fig. 3. We scale $G(u)$ by c_0 and, as shown, subtract a scaled shifted version of $G(u)$. The scaling is chosen so that this difference will totally eliminate the first order spectra initially causing positive frequency aliasing. Thus

$$F(u) = \frac{1}{c_0^2 - c_1 c_{-1}} \left[c_0 G(u) - c_1 G\left(u - \frac{1}{T}\right)\right]; \quad 0 \leq u \leq W. \tag{4}$$

If $f(x)$ is real, knowledge of this portion of the spectrum is

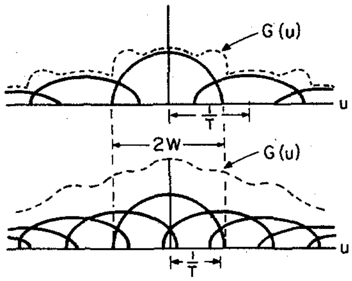

Fig. 2. Illustration of (a) first order aliasing; (b) second order aliasing.

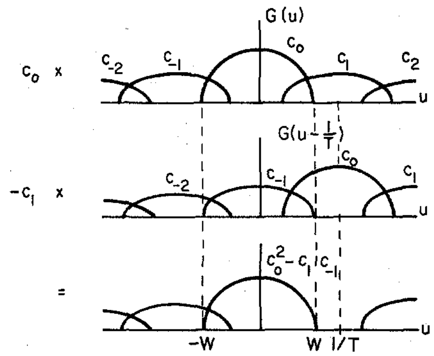

Fig. 3. Removing the first order spectrum by subtracting two weighted and shifted versions of the degraded spectrum.

sufficient to restore the signal since the Fourier transform of a real function is Hermitian:

$$F(u) = F^*(-u).$$

We can then show

$$f(x) = 2 \text{ Re} \int_0^W F(u) \exp(j2\pi ux)\, du$$

where Re denotes "the real part of." Substituting (4) and "simplifying" gives

$$f(x) = \frac{2}{c_0^2 - c_1 c_{-1}} \text{ Re } [g(x) \{c_0 - c_1 \exp(j2\pi x/T)\}] * [W \operatorname{sinc}(Wx) \exp(j\pi Wx)].$$

Note that $g(x)$ is multiplied by a periodic function and filtered. The corresponding restoration algorithm pictured in Fig. 4 follows as

$$f(x) = \frac{2}{c_0^2 - c_1 c_{-1}} \left[\left\{g(x)\left[c_0 \cos \pi Wx - c_1 \cos \pi \left(W - \frac{2}{T}\right)x\right]\right\} * W \operatorname{sinc} Wx\right] \cos \pi Wx + \frac{2}{c_0^2 - c_1 c_{-1}} \left[\left\{g(x)\left[c_0 \sin \pi Wx - c_1 \sin \pi \left(W - \frac{2}{T}\right)x\right]\right\} * W \operatorname{sinc} Wx\right] \sin \pi Wx.$$

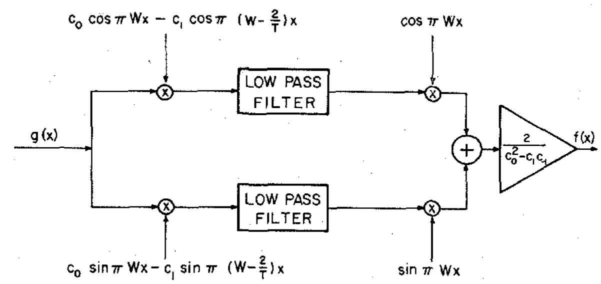

Fig. 4. Restoration of first order aliased data from the shift method illustrated in Fig. 3 using Hermitian symmetry. The filters are unity for $|u| \leqslant W/2$ and zero elsewhere.

A General Interpolation Technique

For Mth order aliasing, there are $2M$ unwanted spectra interfering with the desired zero order spectrum. In this section, we demonstrate a general method for eliminating the unwanted spectra by application of the technique presented in the previous section.

Consider Fig. 5 in which $2M + 1$ shifted versions of $G(u)$ are shown, i.e.,

$$\{G(u - m/T) | M = -M, -M + 1, \cdots, M\}.$$

The interfering component spectra in each shifted G are shown not overlapping for presentation clarity. We now simply need to weight the mth shifted G by a coefficient b_m so that

$$\sum_{m=-M}^{M} b_m G\left(u - \frac{m}{T}\right) \operatorname{rect}\left(\frac{u}{2W}\right) = F(u). \tag{5}$$

With attention again to Fig. 5, this is equivalent to summing the weights of the component spectra in each column to give zero for the interferring spectra and unity for the zero order spectra. That is, find the b_m's which satisfy

$$\sum_{m=-M}^{M} b_m c_{n-m} = \delta_n; \; |n| \leqslant M \tag{6}$$

where δ_n denotes the Kronecker delta. Viewing this as a matrix operation:

$$\begin{bmatrix} c_0 & c_{-1} & \cdots c_{-M} & \cdots c_{-2M} \\ c_1 & c_0 & \cdots c_{-M+1} & \cdots c_{-2M+1} \\ \vdots & \vdots & \vdots & \vdots \\ c_M & c_{M-1} & \cdots c_0 & \cdots c_{-M} \\ \vdots & \vdots & \vdots & \vdots \\ c_{2M} & c_{2M-1} & \cdots c_M & \cdots c_0 \end{bmatrix} \begin{bmatrix} b_{-M} \\ b_{-M+1} \\ \vdots \\ b_0 \\ \vdots \\ b_M \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \tag{7}$$

it is clear the b_m's can be solved for by solution of a Toeplitz set of equations [21].

Inverse transforming (5) gives the spatial domain restoration formula

$$f(x) = [g(x)\, \theta_M(x/T)] * 2W \operatorname{sinc} 2Wx \tag{8}$$

where $\theta_M(x)$ is the trigonometric polynomial

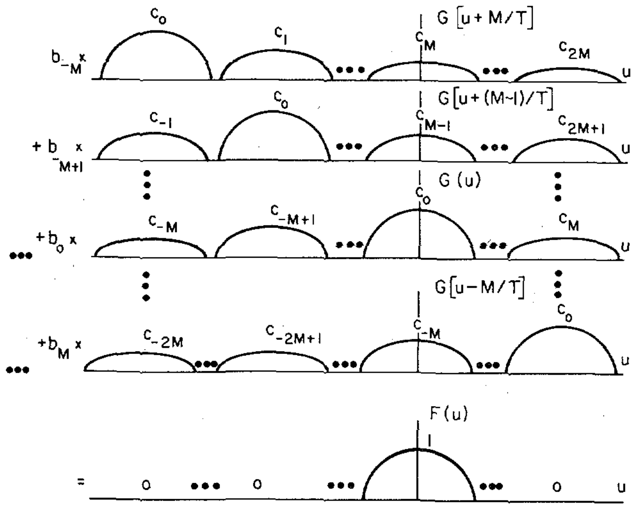

Fig. 5. Illustration of the methodology of restoring Mth order aliased data by summing $2M + 1$ shifted and weighted versions of the degraded spectrum.

$$\theta_M(x) = \sum_{m=-M}^{M} b_m \exp(-j2\pi m x). \tag{9}$$

Note, however, that since

$$g(x) = g(x)\, r_\alpha(x/T)$$

we only require knowledge of θ_M where r_α is unity. Thus we define the periodic function

$$\psi_M(x) = \theta_M(x)\, r_\alpha(x). \tag{10}$$

Expanding in a Fourier series gives

$$\psi_M(x) = \sum_{n=-\infty}^{\infty} d_n \exp(j2\pi n x).$$

The coefficients are

$$\begin{aligned} d_n &= \int_{-1/2}^{1/2} \psi_M(x) \exp(-j2\pi n x)\, dx \\ &= \int_{-\alpha/2}^{\alpha/2} \theta_M(x) \exp(-j2\pi n x)\, dx \\ &= \alpha \sum_{m=-M}^{M} b_m \operatorname{sinc} \alpha(n-m) \end{aligned}$$

where, in the last step we have used (9). From (6), we conclude

$$d_n = \begin{cases} \delta_n; & |n| \leq M \\ \alpha \displaystyle\sum_{m=-M}^{M} b_m \operatorname{sinc} \alpha(n-m); & |n| > M. \end{cases}$$

Note that the d_n's are also the weights of the remaining spectra after restoration. Plots of $\psi_M(x)$ for $\alpha = 0.5$ are shown in Fig. 6. Plots of $\psi_2(x)$ for various duty cycles are shown in Fig. 7.

In lieu of (8), the restoration algorithm pictured in Fig. 8 now becomes

$$f(x) = [g(x)\, \psi_M(x/T)] * 2W \operatorname{sinc} 2Wx. \tag{11}$$

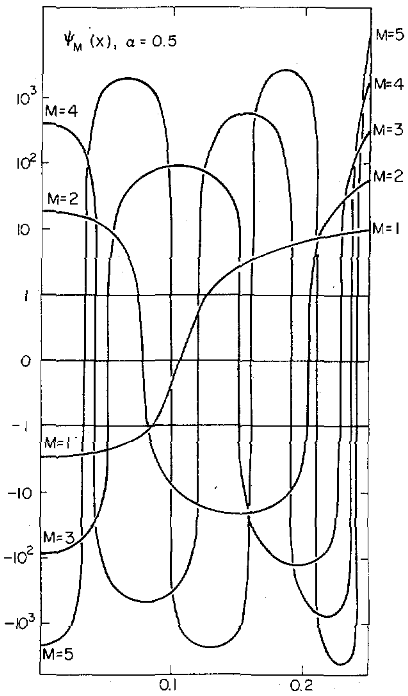

Fig. 6. Plots of $\psi_M(x) = \psi_M(-x)$ for $\alpha = 0.5$ and $M = 1, 2, 3, 4$, and 5. The vertical scale is linear for $|\psi_M| \leq 1$ and logarithmic otherwise.

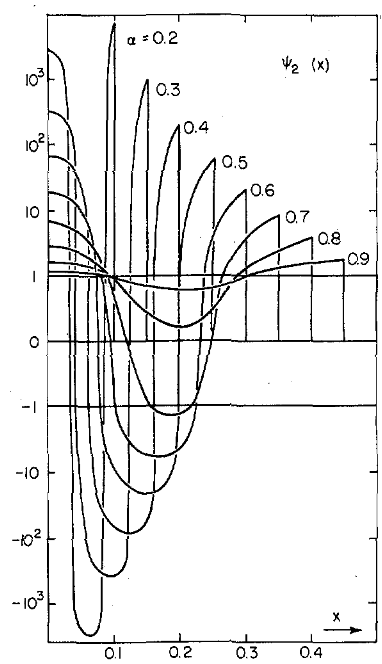

Fig. 7. Plots of $\psi_2(x)$ for various α. The vertical scale is linear for $|\psi_2| \leq 1$ and is logarithmic otherwise.

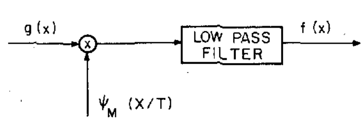

Fig. 8. Restoration of Mth order aliased data. The periodic function, $\psi_M(x/T)$, is parameterized by the degradation duty cycle α, degradation period T, and the order of aliasing M. The filter has bandwidth $2W$.

Notes

1) Taking the limit at $T \to \infty$ holding αT constant gives the classic band-limited signal extrapolation problem [1]-[3]. This problem is ill posed [12], [22], [23]. We thus expect greater and greater noise sensitivity as $\alpha \to 0$.

2) If $N \geqslant M$, then θ_N could be used in (8) in lieu of θ_M and ψ_N in lieu of ψ_M can be used in (11). The restoration algorithm then simply eliminates spectra which do not require elimination. The obvious question of the existence of an $\theta_\infty(x)$ or a $\psi_\infty(x)$ for arbitrary aliasing restoration arises. Consider, however, the product in (10) as $N \to \infty$. The Fourier coefficient of ψ_∞ is given by the discrete convolution the Fourier coefficients of θ_∞ and r_α

$$d_n = \sum_{m=-\infty}^{\infty} b_m c_{n-m}.$$

From (6), $d_n = \delta_n$. It follows that $\psi_\infty(x) \to 1$ and $\theta_\infty(x) \to 1/r_\alpha(x)$. Thus, in the limit as $M \to \infty$, $\theta_M(x)$ becomes unbounded over those intervals where $r_\alpha(x)$ is zero.

3) The sensitivity and instability noted in items (1) and (2) manifest themselves in the algorithm through a worsening condition of the C matrix in (7). That is, as $\alpha \to 0$ or $M \to \infty$, the C matrix becomes more and more ill conditioned [23].

4) The restoration algorithm in (8) is applicable to any periodic degradation of $f(x)$ (under one condition). Simply use the Fourier coefficients of the periodic degradation as the c_n's in (6) to determine the b_m's. The condition is that the corresponding matrix in (7) is not singular. Such is the case, for example, in the Whittaker-Shannon sampling theorem [3], [24] where the periodic degradation is

$$\sum_{p=-\infty}^{\infty} \delta(t - pT).$$

The corresponding c_n's are all equal.

5) Consider replacing $r_\alpha(x/T)$ by $1 - r_\alpha(x/T)$. The restoration algorithm then becomes a possible continuous solution to the classic interpolation problem of recovering $f(x)$ from $f(x)\ [1 - \text{rect}\ \{x/(1-\alpha)\,T\}]$. Part of the known data, however, is obviously not used. Note also here that as $T \to \infty$ with $(1-\alpha)\,T$ held constant, we obtain the classic interpolation problem.

6) Since $c_n = c_{-n}$ in (7), it follows that $b_n = b_{-n}$. Thus, (6) simplifies to

$$b_0 c_n + \sum_{m=1}^{M} b_m (c_{n-m} + c_{n+m}) = \delta_n; \qquad 0 \leqslant n \leqslant M.$$

The order of the corresponding matrix to be inverted for finding the b_m's is thus reduced from $2M + 1$ to $M + 1$. Then, from (9)

$$\theta_M(x) = b_0 + 2 \sum_{m=1}^{M} b_m \cos 2\pi m x.$$

Alternately, we could eliminate only those spectra overlapping the right half of the zero order spectrum and, as before, reconstruct the image from Hermitian symmetry. For Mth order aliasing, M spectra need to be eliminated to the right of the zero order spectra and $\langle M/2 \rangle$ to the left. The corresponding matrix is thus of order $M + \langle M/2 \rangle + 1$. If we further take advantage of the fact that $b_n = b_{-n}$, the matrix reduces to order $M + 1$, identical in dimension to our previous result.

7) One of the referees kindly pointed out a similarity between our algorithm and a technique for eliminating cross-talk over a linear time-invariant channel [25]. Quoting from the review:

"Let $p(t)$ denote a single finite-duration pulse and suppose that the channel response to $p(t)$ as input is $h(t)$. It is assumed that $h(t)$ is known and essentially time limited. For any choice of constants c_r, $r = 0 \to \infty$, and prescribed sampling interval $T > 0$,

$$g(t) = \sum_{r=0}^{\infty} c_r h(t - rT)$$

is the channel response to the input

$$f(t) = \sum_{r=0}^{\infty} c_r p(t - rT).$$

Over the first interval $0 \leqslant t \leqslant T$, $g(t) = c_0 h(t)$; hence, c_0 is uniquely determined. Over the second interval $T \leqslant t \leqslant 2T$,

$$g(t) - c_0 h(t) = c_1 h(t - T)$$

so that c_1 is determined, etc."

Acknowledgment

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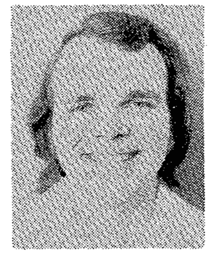

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Fast Algorithms for Linear Prediction and System Identification Filters with Linear Phase

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***Abstract*–A general finite impulse response (FIR) filter can be used as a linear prediction filter, if given only an input sample sequence, or as a system identification model, if given the input and output sequences from an unknown system. With known correlation, the coefficients of the FIR filter that minimize the mean square error in both applications are found by solution of a set of normal equations with Toeplitz structure. Using only data samples, the coefficients that yield the least squared error in both applications are found by solution of a set of normal equations with near-to-Toeplitz structure. Computationally efficient (fast) algorithms have been published to solve for the coefficients from both types of normal equation structures. If the FIR filter is constrained to have a linear phase, then the impulse response must be symmetric. This then leads to normal equations with Toeplitz-plus-Hankel or near-to-Toeplitz-plus-Hankel structure. Fast algorithms for solving these normal equations for the filter coefficients are developed in this paper. They have computational complexity proportional to M^2 and parameter storage proportional to M, where M is the filter order. An application of one of these algorithms for spectral estimation concludes the paper.**

Introduction

The finite impulse response (FIR) filter has served as the foundation for linear prediction signal analysis and has also frequently been used as a system identification model. This paper presents four fast algorithms for linear prediction and system identification when the FIR filter is specialized to have linear phase.

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When used for linear prediction analysis, the FIR filter output represents an estimate of the current input sample value in terms of a linearly weighted sum of past (or future) sample values. If the autocorrelation function for the input process is known, the FIR filter coefficients that will yield the minimum mean square linear prediction error (MMSE) are determined by solving a set of normal equations with Toeplitz structure. A computationally efficient algorithm for solving these normal equations is the Levinson recursion [1], which is one of the most well known fast algorithms in digital signal processing. If one only uses available data samples, the FIR filter coefficients that will yield the least squared linear prediction error (LSE) are determined by solution of a set of normal equations with structure that is near-to-Toeplitz in some sense. A fast algorithm also exists for solving these normal equations [2].

When used for a system identification application, the FIR filter output represents an estimate of the current *output* sample value from an unknown system in terms of a linearly weighted sum of past and/or future sample values from the *input* to the unknown system. If the autocorrelation function of the input process and the cross correlation function between the input and output process are known, the FIR filter coefficients that will minimize the mean square error between

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